Surfaces and Congruences Derived from the Cubic Variety Having a Double Line in Four-Dimensional Space.

By Virgil Snyder.

Twenty years ago Professors Castelnuovo* and Segre† published a series of memoirs on the cubic varieties of ∞^3 points in space of four dimensions S_4 which have been the foundation of many important researches since.‡

The most important ideas are those of the apparent contour (section of the enveloping cone from a given point by ordinary space) and the systems of lines contained upon it which project into bitangents of the apparent contour. The treatment in Segre's longer paper is entirely synthetic; many particular cases are mentioned, but they are not considered in detail. The purpose of the present paper is to extend the results given in the second part of Segre's memoir and to show the connection with a number of known configurations. Part of the results of every article of the present paper except the last are given by Segre, but the method is analytic and no knowledge of the previous papers will be assumed.

§1. *One Double Line.*

1. The most general cubic variety Γ in S_4 having a double line d defined by $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ may be expressed in the form

$$\Gamma \equiv ax_1^2 + hx_1x_2 + bx_2^2 + gx_1x_3 + fx_2x_3 + cx_3^2 = 0, \tag{1}$$

* "Sopra una congruenza del 3° ordine e 6ª classa dello spazio a quattro dimensioni e sulle sue proiezioni nello spazio ordinario," *Atti del Istituto Veneto*, Ser. 6, Vol. V (1887), pp. 1249–1281, and Vol. VI (1888), pp. 525–579.

† "Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario," *Memorie di Torino*, Ser. 2, Vol. XXXIX (1889), pp. 1–48, and "Sulla varietà cubica con dieci punti doppi dello spazio a quattro dimensioni," *Atti di Torino*, Vol. XXII (1887), pp. 791–801.

‡ Particularly by G. Fano, in the *Atti Ist. Veneto* (1896), and *Atti di Torino* (1904); A. Dragoni, in *Batt. Giornale* (1902); and H. W. Richmond, in *Quar. Jour.* (1903).

in which $a \equiv \Sigma a_i x_i$, and similarly for the other coefficients. The tangent S_3 at any point as $(0, 0, 0, t_4, t_5)$ on d is indeterminant. The second polar M_2 has the equation

$$t_4 (a_4 x_1^2 + \dots) + t_5 (a_5 x_1^2 + \dots) = 0, \tag{2}$$

which always contains d. When the point describes d, M_2 forms a pencil, the four basis planes passing through d. Three polars of the pencil break up into pairs of spaces, corresponding to those values of $t_4 : t_5$ for which the discriminant of (2) vanishes. These points are bispatial on d. The six associated tangent spaces define by proper pairs the basis planes of the pencil.

A general S_3 will cut Γ in a cubic surface having a double point where S_3 cuts d. Through this point pass six lines lying on the surface, which contains 15 other lines. The plane formed by joining any point P on Γ to d will cut Γ in d counted twice, and in a line through P. The lines of Γ cutting d define a (1, 6) system. The S_3 tangent to Γ at P will have a double point at P and another at the intersection of S_3 with d. The line joining them is counted twice, hence through P pass 4 lines not cutting d. The residual system of lines on Γ not meeting d is (4, 15).

The apparent contour from a point on Γ will have d for a double line; it will also have a double plane containing four other double points on its conic of contact. The surface F_4 is focal for a (2, 6) congruence having a focal line, and a congruence (8, 15). Any plane through the projection of d will cut F_4 in a conic. The two tangents to this conic from any point of its plane are the lines of the (2, 6) congruence. Any plane π not containing d will cut d in a point Π. Through Π in π can be drawn six tangents to six different conics of the system. On d are three cusps, images of the three bispatial points in S_4, and a fourth, image of the point of intersection of d with the polar S_3 of the center of projection.

When the center of projection is not on Γ, the apparent contour will be a surface F_6 of order 6, class 12, with a cuspidal sextic c_6 and a double line. The congruence of lines tangent to the surface and meeting d is (3, 6). Any plane through d cuts F_6 in a quartic curve having three cusps. The point Π is now a uniplanar triple point on F_6. It is a double point on c_6, with d for a nodal tangent. If the center of projection is in one of the tangent planes touching Γ along d, the cubic covariant G is a ruled surface having d for double directrix. The Hessian will also contain d; hence, from the equation

$$F_6 \equiv 4H^3 + G^2 = 0,$$

d is a triple line on the surface. The residual cuspidal curve is a rational c_4 lying on H, having d for trisecant. Any plane through d cuts F_6 in a cuspidal cubic. When the simple directrix of G lies on H, c_4 breaks up into this directrix and three generators. In this case

$$\Gamma \equiv x_3^3 + x_1 x_2 x_4 + x_5 (x_1^2 + x_2^2 - x_3^2) = 0.$$

The four singular tangent planes are $x_1 = 0$, $x_2 \pm x_3 = 0$; $x_2 = 0$, $x_1 \pm x_3 = 0$; and the three bispatial points on d are $(0, 0, 0, 1, 0)$, $(0, 0, 0, 2, 1)$, $(0, 0, 0, 2, -1)$. The apparent contour from $(0, 1, 1, 0, 0)$ on its polar S_3 or $x_3 = 0$ is

$$27 [x_1 x_2 x_4 + x_5 (x_1^2 + x_2^2)]^2 + 4 [x_1 x_4 + 2 x_2 x_5]^3 = 0.$$

The triple line is $x_1 = 0$, $x_2 = 0$; and the four cuspidal lines are $x_4 = 0$, $x_5 = 0$; $x_1 = 0$, $x_5 = 0$; $x_1 + x_2 = 0$, $x_4 = 2x_5$; $x_1 - x_2 = 0$, $x_4 + 2x_5 = 0$. Of these the first is the simple directrix and the last two are torsal generators of G. The triple line is inflexional tangent for the cubic curve in every plane through it.

2. If the entire plane $x_1 = 0$, $x_2 = 0$ lies on Γ, then $c \equiv 0$. The point $x_1 = 0$, $x_2 = 0$, $f = 0$, $g = 0$ is now a double point on Γ. Let the plane be π_1 and the double point D_1. The pencil $x_2 = kx_1$ through π_1 will cut Γ in π_1 and a pencil of quadrics

$$(a + 2hk + bk^2)\, x_1 + 2x_3 (kf + g) = 0$$

which contains d and a variable line passing through D_1. Conversely, if Γ contains a double point not lying on d, the plane containing it and d will lie entirely on Γ.

The plane $x_1 = 0$, $x_2 = 0$ is now one of the basis planes touched by all the tangent hypercones having the entire line d as vertex and belonging to the points of d. Such hypercones are called hypercones of the second species.

The lines of Γ form three systems: the first $(1, 5)$ composed of the lines cutting d; the second $(1, 5)$ composed of the lines cutting π_1, but not cutting d; and a $(3, 10)$ cutting neither. The second consists of the generators of the quadrics, of the same system as d, while the first is composed of the generators of the other system.

The contours F_4, F_6 are now of class 10, have a double point, and are focal surfaces of congruences $(2, 5)$, $(3, 5)$.

3. If Γ also contains the plane $x_1 = 0$, $x_3 = 0$, or π_2, then $b \equiv 0$. We may write $f \equiv x_4$, $g \equiv x_5$ without loss of generality. The point $h = 0$, $x_4 = 0$ or D_2 is also a double point. The space $x_1 = 0$ cuts Γ in the planes π_1, π_2 and

the plane $x_4 = 0$ or σ_3, passing through both double points and meeting d in $(0, 0, 0, 0, 1)$. The pencil $kx_1 = x_4$ through π_2 cuts Γ in the system of quadrics, whose section by $x_1 = 0$ is the pencil of conics

$$x_3 x_5 + (h_2 x_2 + h_3 x_3 + h_5 x_5) x_2 + k x_2 x_3 = 0.$$

These conics all pass through D_1, D_2 and touch each other on d, the common tangent being $h_5 x_2 + x_3 = 0$. It is a bispatial point on d.

The lines of Γ are now the $(1, 4)$ meeting d, the $(1, 4)$ meeting π_1 but not d, the $(1, 4)$ meeting π_2, the $(2, 6)$ meeting σ_3. Γ can be generated by the trilinear system

$$\begin{aligned} &\alpha x_1 - \beta x_3 = 0, \\ &\alpha x_5 + \beta \bar{a} + \gamma x_2 = 0, \\ &\alpha x_4 + \beta(a_2 x_1 + h) - \gamma_1 = 0, \end{aligned}$$

and its conjugate, where $\bar{a} = a - a_2 x_2$.

4. If Γ also contains the plane $x_2 = 0$, $x_3 = 0$ or π_3, then $a \equiv 0$. There is also a double point D_3 at $h = 0$, $x_5 = 0$ in π_3. The variety now contains six planes π_1, π_2, π_3, σ_1, σ_2, σ_3 and three double points D_1, D_2, D_3. Each plane σ_i cuts d in a bispatial point and contains two double points. Γ has just one proper tangent plane touching it throughout d.

The lines of Γ are arranged in five systems, all of form $(1, 3)$. The first is composed of the lines cutting d, the next three cut $\pi_i \sigma_i$, and the last cuts $\sigma_1 \sigma_2 \sigma_3$.

From a point P on Γ the apparent contour is a surface of order 4, class 6, having a double line with four cuspidal points, four double planes with four double points besides the intersection of each with d, and three other double points. It is complete focal surface for four $(2, 3)$ congruences, and the double line is focal line for another, the surface being the other sheet of the focal surface.

From a point not on Γ the apparent contour will be of order and class 6, having a cuspidal sextic, a double line and three other double points. If the point be taken in the plane $D_1 D_2 D_3 = \tau$, the double points will be collinear. The only proper tangent plane containing d is

$$h_4 x_1 + x_3 = 0, \quad h_5 x_2 + x_3 = 0.$$

The equations of τ are

$$h_1 x_1 + h_4 x_4 = 0, \quad h_2 x_2 + h_5 x_5 = 0.$$

From the point of intersection of these two planes $(-h_5^2 h_4, -h_4^2 h_5, h_4^2 h_5^2, h_4^2 h_2, h_5^2 h_1)$ F_6 will have a triple line and three collinear double points. If the center of projection be taken in τ on Γ, a double point becomes uniplanar, and one conic of contact in a double plane is two coincident lines.

5. Finally, it is possible that

$$\Gamma \equiv h x_1 x_2 + x_2 x_3 x_4 + x_1 x_3 x_5 = 0$$

contain another plane passing through d. Its equations may be taken

$$x_1 = x_2 = x_3.$$

It will lie on Γ if $h_4 = -1$, $h_5 = -1$, $h_1 + h_2 + h_3 = 0$, and there will be a double point $(1, 1, 1, h_1, h_2)$ or D_4 lying in π_4.

$$\Gamma \equiv x_2 (x_1 - x_3)(h_1 x_1 - x_4) + x_1 (x_2 - x_3)(h_2 x_2 - x_5) = 0. \tag{3}$$

The four double points are:

$$D_1 \equiv (0, 0, 1, 0, 0), \qquad D_3 \equiv (1, 0, 0, h_1, 0),$$
$$D_2 \equiv (0, 1, 0, 0, h_2), \qquad D_4 \equiv (1, 1, 1, h_1, h_2);$$

they all lie in the plane τ

$$x_5 - h_2 x_2 = 0, \quad x_4 - h_1 x_1 = 0,$$

which from (3) lies on Γ.

Besides the planes τ, π_i, Γ contains 6 planes σ_{ik} defined as the residual intersection with Γ of the space containing the planes π_i, π_k. The equations of the complete set of eleven planes are:

$$\begin{array}{lrl}
\pi_1: & x_1 = 0, & x_2 = 0. \\
\pi_2: & x_1 = 0, & x_3 = 0. \\
\pi_3: & x_2 = 0, & x_3 = 0. \\
\pi_4: & x_1 - x_3 = 0, & x_2 - x_3 = 0. \\
\sigma_{12}: & x_1 = 0, & x_4 = 0. \\
\sigma_{13}: & x_2 = 0, & x_5 = 0. \\
\sigma_{14}: & x_1 - x_2 = 0, & h_1 x_1 + h_2 x_2 - x_4 - x_5 = 0. \\
\sigma_{23}: & x_3 = 0, & h_1 x_1 + h_2 x_2 - x_4 - x_5 = 0. \\
\sigma_{24}: & x_1 - x_3 = 0, & h_2 x_2 - x_5 = 0. \\
\sigma_{34}: & x_2 - x_3 = 0, & h_1 x_1 - x_4 = 0. \\
\tau\ : & h_1 x_1 - x_4 = 0, & h_2 x_2 - x_5 = 0.
\end{array}$$

The planes π_i contain each one double point D_i and d; the planes σ_{ik} contain two double points D_i, D_k and cut d in bispatial points; the plane τ

passes through all four double points and does not meet d. Thus, through each point D_i pass five planes τ, π_i, σ_{ih}. $(h \neq i.)$

6. The variety Γ may be generated by the following trilinear systems:

$$\mathrm{I}\begin{cases} \alpha x_2 + \beta(x_2 - x_3) = 0, \\ \beta(x_1 - x_3) + \gamma(h_2 x_2 - x_5) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha x_1 = 0. \end{cases} \quad \mathrm{II}\begin{cases} \alpha x_2 + \beta(h_2 x_2 - x_5) = 0, \\ \beta(x_1 - x_3) + \gamma x_1 = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha(x_2 - x_3) = 0. \end{cases}$$

$$\mathrm{III}\begin{cases} \alpha x_2 + \beta x_1 = 0, \\ \beta(x_1 - x_3) + \gamma(h_2 x_2 - x_5) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha(x_2 - x_3) = 0. \end{cases} \quad \mathrm{IV}\begin{cases} \alpha x_2 + \beta(h_2 x_2 - x_5) = 0, \\ \beta(x_1 - x_3) + \gamma(x_2 - x_3) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha x_1 = 0. \end{cases}$$

$$\mathrm{V}\begin{cases} \alpha x_2 + \beta x_1 = 0, \\ \beta(x_1 - x_3) + \gamma(x_2 - x_3) = 0, \\ \gamma(h_1 x_1 - x_4) + \alpha(h_2 x_2 - x_5) = 0. \end{cases}$$

The system V consists of lines cutting d and τ. The pencil of spaces through τ will cut d in a projective range. The quadric surface in the residual section with Γ will have a double point on d; hence every surface of the system will be a cone. In case of the three spaces through τ:

$$\begin{aligned} h_1 x_1 - x_4 &= 0, \\ h_2 x_2 - x_5 &= 0, \\ h_1 x_1 - x_4 - (h_2 x_2 - x_5) &= 0, \end{aligned}$$

the quadric cone breaks up into a pair of planes. These are

$$\begin{matrix} \sigma_{23}, & \sigma_{14}, \\ \sigma_{13}, & \sigma_{24}, \\ \sigma_{12}, & \sigma_{34}, \end{matrix}$$

respectively. The points on d on the lines of intersection of these planes are bispatial points:

$$\begin{array}{lll} (0, 0, 0, 0, \ \ 1), & \text{with the tangent spaces} & x_1 = 0,\ x_2 - x_3 = 0, \\ (0, 0, 0, 1, \ \ 0), & \text{" \ " \ " \ "} & x_2 = 0,\ x_1 - x_3 = 0, \\ (0, 0, 0, 1, -1), & \text{" \ " \ " \ "} & x_3 = 0,\ x_1 - x_2 = 0, \end{array}$$

respectively. The four pencils $\pi_i D_i$ do not belong to V.

The lines of I cut π_1, σ_{24}, σ_{23}, σ_{34}. A pencil of spaces through π_1 will cut Γ in a series of quadrics, whose section by π_1 is d and a line through D_1. The lines of I do not cut d, hence belong to the same system of generators of these

quadrics as d. In general the pencils σ_{ik}, D_k belong to (i). There are no proper quadric cones on Γ with vertices at D_i. The sextic cones at the double points break up into plane pencils. All of these systems are (1, 2).

7. The apparent contour from a point on Γ will be a surface of order and class 4, having a double line, two double planes and four double points. It is the general complex surface of Plücker, formed by the lines of a quadratic complex which cut a given line not belonging to the complex. Its symbol is [21111].*

If, however, the center of projection be taken in τ, all the four double points will project into collinear points, and the line joining them will be a double line. The lines of V now project into lines cutting two skew double lines, hence F_4 is a ruled surface contained in a linear congruence. From $(1, 2, 3, h_1, 2h_2)$ on $x_5 = 0$ the contour becomes

$$[2(x_1 - x_2 - x_3)(h_1 x_1 - x_4) + h_2 x_2 (x_2 - x_3 - x_1)]^2$$
$$+ 4[x_2(x_1 - x_3)(h_1 x_1 - x_4) + h_2 x_1 x_2 (x_2 - x_3)]\,[4(h_1 x_1 - x_4) + h_2 x_2] = 0.$$

The double directrices are $x_3 - 3x_1 = 0$, $x_2 - 2x_1 = 0$; $x_2 = 0$, $h_1 x_1 - x_4 = 0$. The symbol is [(11)1111].

8. The apparent contour from a point not on Γ is a surface of order 6, class 4, having a double line, a cuspidal sextic curve and four double points. An interesting case arises when $h_1 = h_2 = 0$. If we project from $(0, 1, 1, 1, 0)$ on the polar space $x_1 = x_2 + x_3 + x_4$ the equation of the focal surface becomes

$$4[(x_2 + x_4)^2 - x_2 x_4]^3 = 27[x_2 x_4 (x_2 + x_4) + x_1 x_5 (2x_2 - x_1 + x_4)]^2.$$

The double line becomes $x_1 = 0$, $2x_2 + x_4 = 0$, and the cuspidal curve is composed of the two plane cubics

$$x_2 = \omega x_4, \qquad x_2^3 = x_1 x_5((2 + \omega) x_2 - x_1), \qquad [\omega^3 = 1$$

each of which has a node at $(x_1, x_2, x_4, x_5) \equiv (0, 0, 0, 1)$ on the double line. Any plane section through the double line will consist of a quartic having three cusps, one of which is at $(0, 0, 0, 1)$. There are but five distinct double planes:

$$\begin{aligned}
\pi_1 &= \pi_2 = \sigma_{12} \equiv x_1 = 0, \\
\pi_3 &= \pi_4 = \sigma_{34} \equiv 2x_2 - x_1 + x_4 = 0, \\
\sigma_{13} &= \sigma_{24} = \tau \equiv x_5 = 0, \\
\sigma_{14} &\equiv x_1 - x_2 + x_4 + x_5 = 0, \\
\sigma_{23} &\equiv x_2 + 2x_4 + x_5 - x_1 = 0.
\end{aligned}$$

*A. Weiler: "Ueber die verschiedenen Gattungen der Complexe zweiten Grades," *Math. Ann.*, Vol. VII (1874), or the corrected list given by Jessop, *Treatise on the Line Complex*, p. 230.

Three double planes pass through each double point. The sections made by the double planes σ_{12}, σ_{34}, τ consist of three concurrent lines, each counted twice. Thus, through $(0, 0, 0, 1)$ pass the double line and four simple lines of the surface, the latter having fixed tangent planes throughout.

The bitangents of the surface can be readily arranged in four systems of quadrics, each of index 3. Thus III becomes

$$\frac{\alpha\beta(2x_2+x_4)+\beta^2x_1}{\alpha x_5}=\frac{\alpha^2(x_1-2x_2-x_4)}{\alpha(x_2-x_4)+\beta x_1}.\text{*}$$

§ 2. *Two Intersecting Double Lines.*

9. The variety Γ may have the line d' defined by $x_1=0$, $x_2=0$, $x_4=0$ for double line, as well as d. The general equation is

$$\Gamma \equiv ax_1^2+hx_1x_2+bx_2^2+x_1x_3x_4=0.$$

The tangent S_3 to Γ at any point $(0, 0, \xi_3, \xi_4, \xi_5)$ lying in the plane $x_1=0$, $x_2=0$ or π has the equation $x_1=0$. The residual intersection of $x_1=0$ with Γ is the plane σ defined by $b=0$, cutting d in the point $(0, 0, 0, -b_5, b_4)$ and d' in the point $(0, 0, b_5, 0, -b_3)$. The polar M_2 of the first point is

$$b_5(a_4x_1^2+h_4x_1x_2+b_4x_2^2+x_1x_3)-b_4(a_5x_1^2+h_5x_1x_2+b_5x_2^2)=0,$$

which factors into x_1 and another linear factor; hence the point is bispatial. The point on d' is also bispatial, $x_1=0$ being common to both. The $M_2^{(3)}$ of $(0, 0, 0, 0, 1)$, the point of intersection of d, d', is $a_5x_1^2+h_5x_1x_2+b_5x_2^2=0$. It consists of two spaces, both passing through π. The spaces of the pencil $x_2=kx_1$ passing through π cut Γ in a series of quadrics

$$x_1(a+kh+k^2h)+x_3x_4=0$$

cutting π in d, d'. The systems of lines cutting d, d' are each $(1, 4)$.

A pencil $x_1=\lambda b$ through σ cuts Γ in a series of quadrics whose section with σ is a pencil of conics touching each other at the bispatial points on d, d'. This system is $(2, 6)$, the class being obtained as follows. Any S_3 cuts Γ in a cubic surface having two double points, containing 16 lines made up of the line joining the two double points, 4 others through each, and 7 not passing through either. Of the latter, one is the intersection with σ.

* For a similar particular case of a variety having nine distinct double points see my paper in the *Trans. Amer. Math. Soc.*, Vol. X (1909). The necessary and sufficient condition that the cuspidal sextic breaks up into two plane cubics is that the Hessian is the product of two spaces. This will happen when the center of projection lies on a definite curve of S_4.

10. Now suppose Γ has a double point not lying in π nor in σ. $D_1 \equiv (1, 0, 0, 0, 0)$. The plane determined by D_1 and d must lie entirely on Γ. This is $x_2 = 0$, $x_3 = 0$; hence $a = 0$. Similarly, the plane of D_1 and d' lies on Γ, but $x_2 = 0$, $x_4 = 0$ already satisfies the equation.

$$\Gamma \equiv hx_1 x_2 + bx_2^2 + x_1 x_3 x_4 = 0.$$

The system of lines cutting d is now $(1, 3)$. Its equations are

$$\begin{aligned} \alpha x_2 - \beta x_3 &= 0, \\ \alpha x_4 + h\beta + \gamma b &= 0, \\ \beta x_2 - \gamma x_1 &= 0. \end{aligned}$$

The $(1, 3)$ system cutting d' has the equations

$$\begin{aligned} \alpha x_2 - \beta x_4 &= 0, \\ \alpha x_3 + \beta h + \gamma b &= 0, \\ \beta x_2 - \gamma x_1 &= 0. \end{aligned}$$

The conjugate of the first system cuts σ and the plane $D_1 d'$. The conjugate of the second cuts σ and the plane $D_1 d$.

11. If Γ has another double point D_2 at $(1, 1, 0, 0, 0)$, and $b \equiv x_5$,

$$\Gamma \equiv (x_1 - x_2) x_2 x_5 + x_1 x_3 x_4 = 0.$$

The plane σ and the line $D_1 D_2$ lie in the space $x_5 = 0$ which cuts Γ in σ and two new planes

$$\tau : x_3 = 0,\ x_5 = 0; \qquad \tau_2 : x_4 = 0,\ x_5 = 0,$$

which intersect in the line $D_1 D_2$. The first passes through the bispatial point on d, the second through the bispatial point on d'. The eight planes on Γ are now the following:

$$\begin{array}{lcr} d,\ d' & : & x_1 = 0,\ x_2 = 0, \\ D_1 d & : & x_2 = 0,\ x_3 = 0, \\ D_1 d' & : & x_2 = 0,\ x_4 = 0, \\ D_2 d & : & x_1 - x_2 = 0,\ x_3 = 0, \\ D_2 d' & : & x_1 - x_2 = 0,\ x_4 = 0, \\ \sigma & : & x_1 = 0,\ x_5 = 0, \\ \tau_1 & : & x_3 = 0,\ x_5 = 0, \\ \tau_2 & : & x_4 = 0,\ x_5 = 0. \end{array}$$

There are now four systems $(1, 2)$:

$$\begin{array}{rcl} \text{I} & \text{cuts} & D_2 d,\ D_1 d',\ \sigma. \\ \text{II} & \text{“} & \tau_1,\ d'. \\ \text{III} & \text{“} & \tau_2,\ d. \\ \text{IV} & \text{“} & D_1 d,\ D_2 d',\ \sigma. \end{array}$$

The first and third are conjugate, as are the second and fourth.

12. The apparent contour from $(0, 1, 0, 0, 1)$ on its polar S_3 is

$$4[3x_2^2 - 3x_1 x_2 + x_1^2]^3 - 27[(x_1 - x_2)(x_1 - 2x_2)x_2 + x_1 x_3 x_4]^2 = 0.$$

It is of order 5, as x_1 appears as a factor. Any plane $x_2 - k\, x_1 = 0$ will cut the surface in the line $x_1 = 0$, $x_2 = 0$ and two conics which touch each other at $(0, 0, 0, 1)$, $(0, 0, 1, 0)$, the common tangents being $x_3 = 0$, $x_4 = 0$. The planes $x_3 = 0$, $x_4 = 0$ are double planes, the conics of contact breaking up into two lines intersecting at the point of contact of the conics, which are both triple points on the surface. The plane $x_1 = 0$ contains the line joining the triple points as a threefold line. The points $(3, 2, 0, 0)$, $(3, 1, 0, 0)$ are both double. The planes $x_1 = k_1 x_2$, $x_1 = k_2 x_2$ cut the surface in conics which are cuspidal on the surface, k_1, k_2 being roots of the quadratic equation $3k^2 - 3k + 1 = 0$.

The planes (dd') and σ project into $x_1 = 0$.
" " $D_1 d$, $D_2 d$, τ_1 " " $x_3 = 0$.
" " $D_1 d'$, $D_2 d'$, τ_2 " " $x_4 = 0$.

The congruences II, III have focal lines; the other two interchange by the transposition $(x_3 x_4)$, an involution which leaves the surface invariant.

13. If the center of projection be taken upon Γ, the apparent contour is the Plücker complex surface with one more double line, having the symbol $[1122]$. The usual form of equation results when the center is at $(2, 4, 2, 2, 1)$, and an interesting particular case from $(1, 0, 0, 0, 1)$.

§ 3. *Three Concurrent Double Lines.*

14. If Γ has three double lines d, d', and d'', the last defined by $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, its equation becomes

$$\Gamma \equiv (ax_3 + bx_4)x_1 x_2 + x_2^2 x_5 + (cx_1 + dx_2)x_3 x_4 = 0.$$

The tangent S_3 at every point of the plane π_{12} containing the first two lines is $cx_1 + dx_2 = 0$. It cuts Γ in the residual plane $\sigma_3 : d(ax_3 + bx_4) - cx_5 = 0$. The tangent S_3 at points of $\pi_{13} : x_2 = 0$, $x_3 = 0$ is $bx_2 + cx_3 = 0$, and the residual plane is $\sigma_2 : b(ax_1 + dx_4) - cx_5 = 0$. For $\pi_{23} : x_2 = 0$, $x_4 = 0$ the tangent is $ax_2 + cx_4 = 0$, and the residual plane is $\sigma_1 : a(bx_1 + dx_3) - cx_5 = 0$. The point $(0, 0, 0, 0, 1)$ is unispatial, $x_2 = 0$ being the tangent S_3. On d', the planes σ_2 and σ_3 meet in the bispatial point $(0, 0, 0, c, bd)$, the tangent spaces being

$$c[bx_1 x_2 + x_3(cx_1 + dx_2)] + dbx_2^2 = 0.$$

The lines of Γ form three $(1, 2)$ systems, cutting d, σ_1; d', σ_2; d'', σ_3.

15. From a point not on Γ the apparent contour becomes an F_6 with a cuspidal c_6, three concurrent double lines and seven triple points. It is the dual of the cubic surface cut from Γ by a tangent S_3. Each of the three (3, 2) congruences has a focal line.

From a point on Γ, the contour becomes the Steiner surface of order 4, class 3, and symbol [222]. By taking the center at (0, 1, 1, 0, 0), the section by $x_2 = 0$ is the canonical form

$$x_1^2 x_3^2 + x_3^2 x_4^2 + x_4^2 x_1^2 - 2x_1 x_3 x_4 (2x_5 - x_3) = 0.$$

If the center of projection be taken in π_{ik}, [42] results.

§ 4. *One Double Line of the Second Species.*

16. A double line on Γ is said to be of the second species when the tangent $M_2^{(3)}$ at every point upon it breaks up into two spaces. If $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ be the double line, then

$$\Gamma \equiv ax_1^2 + hx_1 x_2 + bx_2^2 + Ax_3^2 = 0,$$

$A = \sum_{i=1}^{3} A_i x_i$. The tangent $M_2^{(3)}$ of $(0, 0, 0, \xi_4, \xi_5)$ is

$$\xi_4(a_4 x_1^2 + h_4 x_1 x_2 + b_4 x_2^2) + \xi_5(a_5 x_1^2 + h_5 x_1 x_2 + b_5 x_2^2) = 0.$$

Thus, the pairs of spaces have the plane $x_1 = 0$, $x_2 = 0$ in common, and form an involution projective with the point of tangency. The points on d belonging to the double elements of the involution will be unispatial. The section of Γ made by the space $x_2 = kx_1$ through the basis plane π will be

$$\phi_3(x_1, x_3) + [x_4 (a_4 + h_4 k + b_4 k^2) + x_5 (a_5 + h_5 k + b_5 k^2] x_1^2 = 0,$$

hence a cubic cone with vertex at $(0, 0, 0, a_5 + h_5 k + b_5 k^2, -(a_4 + h_4 k + b_4 k^2))$ having d for cuspidal edge, and π for tangent plane. The lines cutting d form a (1, 6) system; the residual system on Γ is (3, 9).

17. The apparent contour of Γ from a point upon it is a surface of order 4, class 6, with a cuspidal line having a fixed tangent plane and two triple points. The surface has a double plane, the conic of contact passing through two double points besides the cusp on d. It is focal surface of a (2, 6) and a (6, 9) congruence, the former having a focal line.

From a point not on Γ the contour is of order and class 6, having a cuspidal line and a cuspidal sextic with a point in common. This and the projection of the two unispatial points are triple points on the surface. The cuspidal line has a fixed tangent plane which cuts F_6 in three lines meeting at the uniplanar triple

point. In particular, if P be taken on π, its polar $M_2^{(3)}$ will be a hypercone having d for vertex, and its polar S_3 will always pass through d. Its equation is $A + 2A_3x_3 = 0$, independent of P. Hence every line of the (1, 6) system cutting d will go into a line lying entirely on the focal surface, a ruled surface of order 6, having d for multiple directrix. Its equation is of the form

$$[\phi_2(x_1, x_2)]^3 + K[x_5 x_1^2 + x_4 x_2^2 + hx_1 x_2 + \phi_3(x_1, x_2)]^2 = 0.$$

Any plane $x_2 = kx_1$ through the fourfold line contains two generators which intersect upon it, the point being the same for two values of k. The generators passing through the images of the unispatial points are cuspidal. The surface is rational and has no other double line. It is form 101 in my enumeration.* An interesting particular case is that in which the double elements of the involution approach coincidence. The ruled sextic now has a tacnodal generator. Its equation is

$$(x_4 x_2^2 - x_1^2 x_3)^2 = x_1 x_2^5.$$

18. Suppose Γ contains a double point not lying on d. This requires that $A \equiv 0$. Now any space $x_2 = kx_1$ passing through π will contain π as a squared factor, the other being

$$a + hk + bk^2 = 0\,;$$

hence Γ contains an infinite number of planes, each lying in an S_3 with π. Every point $(0, 0, \xi_3, \xi_4, \xi_5)$ of π is bispatial, the tangent M_2 being of the form

$$\xi_3(a_3 x_1^2 + h_3 x_1 x_2 + b_3 x_2^2) + \xi_4(a_4 x_1^2 + \dots) + \xi_5(a_5 x_1^2 + \dots) = 0.$$

The two spaces become coincident for points satisfying the equation

$$4(a_3 \xi_3 + a_4 \xi_4 + a_5 \xi_5)(b_3 \xi_3 + b_4 \xi_4 + b_5 \xi_5) = (h_3 \xi_3 + h_4 \xi_4 + h_5 \xi_5)^2.$$

Thus, *if Γ has a double plane, there is a conic lying in the plane, locus of unispatial points.*

Since a space $x_2 = kx_1$ is uniquely determined by a point not in π, it follows that no two planes of Γ

$$x_2 = kx_1, \qquad a + hk + bk^2 = 0;$$
$$x_2 = lx_1, \qquad a + hl + bl^2 = 0$$

can lie in the same S_3, except when $h = 0$; but in this case Γ is a hypercone, projection of a ruled cubic surface of S_3 from a point outside. Excluding this case, it is seen that through any point $(0, 0, \xi_3, \xi_4, \xi_5)$ of π pass two lines, $a + hk + bk^2 = 0$, intersection of the generating planes with π. The envelope of these lines is the conic of unispatial points c_2.

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* JOURNAL, Vol. XXVII (1905), pp. 76–102. See p. 95.

The tangent space T at a point P of c_2 will cut π in the tangent t to c_2 at P. It will cut Γ in a generator plane α and in a quadric surface. The quadric passes through t and another line of α. The generators of one system will cut t, those of the other not cutting π. There is just one tangent space at each point of c_2, hence Γ contains a $(1, 1)$ system of lines not lying in the generator planes.

19. The apparent contour from a point not on Γ,

$$\Gamma \equiv x_3 x_1^2 + x_4 x_1 x_2 + x_5 x_2^2 = 0,$$

say $(0, 1, 0, 0, 1)$, is

$$4(x_1 x_4 - 3x_2^2)^3 + 27 \left[x_3 x_1^2 + x_1 x_2 x_4 - 2x_2^3\right]^2 = 0,$$

which contains x_1^2 as a factor. The other factor is the envelope of the plane

$$k^3 x_1 - 3k^2 x_2 + kx_4 + x_3 = 0,$$

the projection of a generating plane.

From a point on Γ, the apparent contour consists of π counted twice, the generating plane through the center of projection and the point which is the projection of the transversal through it.

20. In the preceding case, the tangent spaces formed an involution. Now suppose one factor x_3 is fixed while the other space turns about π projective with the point of tangency on d. In the equation of Γ, a, b, h, c can contain only x_1, x_2, x_3, while f, g are general. The equation of the variable tangent S_3 is

$$\xi_4 (g_4 x_1 + f_4 x_2) + \xi_5 (g_5 x_1 + f_5 x_2) = 0.$$

The fixed space $x_3 = 0$ cuts Γ in

$$(a_1 x_1 + a_2 x_2) x_1^2 + (h_1 x_1 + h_2 x_2) x_1 x_2 + (b_1 x_1 + b_2 x_2) x_2^2 = 0;$$

i. e., in three planes π_1, π_2, π_3 passing through d. The section made by any S_3 through π_i, say

$$\alpha_i x_1 + \beta_i x_2 = kx_3,$$

will be a quadric

$$\psi_2 (x_1, x_2) + (f_4 x_4 + f_5 x_5) x_2 + (g_4 x_4 + g_5 x_5) x_1 = 0,$$

whose section with π_i is

$$x_1 (B x_1 + (f_4 x_4 + f_5 x_5) \alpha_i + (g_4 x_4 + g_5 x_5) \beta_i) = 0,$$

thus the line d and a variable line, meeting d in the fixed point of contact of the space $\pi_i \pi$.

21. If Γ has a double point D not lying on d, the plane Dd will lie on Γ, hence D will either lie in π or in $\pi_i = \pi_k$.

Let $\pi_1 = \pi_3$ and P_1 be the point of contact on d of the space $\pi_1 \pi$. A space passing through π_1 will cut Γ in π_1 and a quadric, the section of the latter by π_1 being d and $P_1 D$; hence this line must be a double line. If π_1 be $x_1 = 0$, $x_3 = 0$, then $b_1 = b_2 = h_2 = 0$. The point on d associated with $x_1 = 0$ is $(0, 0, 0, f_6, -f_4)$, and the equations of the new double line are

$$x_1 = 0, \quad b_3 x_2 + f_2 x_2 + f_4 x_4 + f_5 x_5 = 0, \quad x_3 = 0.$$

This is the most general Γ having a double line of the second species and another double line of the first species. There are evidently three (1, 3) systems of lines, one cutting d, one cutting the other double line, and the third cutting π_2.

22. Now suppose $D \equiv (0, 0, \xi_3, \xi_4, \xi_5)$ in π. This requires that $c \equiv 0$, $g_3 = 0$, $f_3 = 0$.

$$\Gamma \equiv \prod_{i=1}^{3} (\alpha_i x_1 + \beta_i x_2) + x_3 (f x_2 + g x_1) = 0.$$

The space $\pi \pi_i$ cuts Γ in $\sigma_i \equiv f \beta_i - g \alpha_i = 0$, which passes through D, and cuts d in the point belonging to $\pi \pi_i$. Γ now has four systems (1, 2), of which the first cuts d, and the others cut π_i, σ_{i+1}, σ_{i+2}.

The general equation of Γ is

$$\Gamma \equiv x_1 x_2 (x_1 - x_2) + x_3 (x_1 x_4 + x_2 x_5) = 0.$$

$$\begin{array}{llll}
\pi: & x_1 = 0, \ x_2 = 0. & \sigma_1: & x_1 = 0, \ x_5 = 0. \\
\pi_1: & x_1 = 0, \ x_3 = 0. & \sigma_2: & x_2 = 0, \ x_4 = 0. \\
\pi_2: & x_2 = 0, \ x_3 = 0. & \sigma_3: & x_1 - x_2 = 0, \ x_4 + x_5 = 0. \\
\pi_3: & x_1 - x_2 = 0, \ x_3 = 0. & &
\end{array}$$

The equations of the four systems are

$$\begin{array}{ll}
\alpha x_2 + \beta x_1 = 0, & \alpha x_2 + \beta x_4 = 0, \\
\alpha x_4 + \gamma (x_2 - x_1) - \beta x_5 = 0, & \beta x_1 - \gamma x_3 = 0, \\
\beta x_1 - \gamma x_3 = 0, & \alpha x_1 - \beta x_5 + \gamma (x_2 - x_1) = 0, \\
\alpha x_1 + \beta x_5 = 0, & \alpha x_2 + \beta x_4 = 0, \\
\alpha x_2 - \beta x_4 + \gamma (x_2 - x_1) = 0, & \alpha x_1 - \beta x_5 + \gamma x_1 = 0, \\
\beta x_2 - \gamma x_3 = 0, & \beta (x_1 - x_2) + \gamma x_3 = 0.
\end{array}$$

23. A double line $(x_1 = 0, x_3 = 0, x_5 = 0)$ and a double point $(0, 0, 1, 0, 0)$ both lie on Γ, when its equation is of the form

$$\Gamma \equiv x_1^2 x_2 + x_3 (x_1 x_4 + x_2 x_5) = 0.$$

In the preceding table, π_3 coincides with π_1, and σ_3 with σ_1. There are therefore three systems of lines (1, 2), the first cutting d, the second d', σ_2, the third π_2, σ_1.

24. In 19–22 the fixed basis plane π did not lie in the fixed tangent space. Now let $x_1 = 0$ be the fixed tangent space, π as before. In this case Γ can have no other double point nor double line. It has four (1, 3) systems.

25. The apparent contour from a point on Γ is of order 4, class 6, except when Γ has a double point, in which case it is of class 4. The general one is [3111], formed by the lines of a quadratic complex which meet a line belonging to the complex. If Γ has the double point and a double line we have [123], wherein d is a line of the complex and is tangent to the Kummer surface which is its surface of singularities. From $(1, 1, 1, -1, 0)$ on $x_1 = 0$ the equation is

$$[x_3 x_4 + x_3 x_5 + x_2 x_5]^2 - 4\,[x_2 - x_3 + x_4 + x_5]\, x_2 x_3 x_5 = 0.$$

The line $x_2 = 0$, $x_3 = 0$ is the cuspidal line, $x_3 = 0$, $x_5 = 0$ the double one. The projection of D is $(0, 1, 0, 0)$. The plane $x_2 = 0$ cuts F_4 in the cuspidal line and touches it along the line $x_4 + x_5 = 0$, projection of σ_1. $x_5 = 0$ passes through the double line and touches the surface along $x_4 = 0$. The plane $x_2 - x_3 + x_4 + x_5 = 0$ touches it along the conic

$$(x_3 - x_5)(x_4 + x_5) + x_3 x_5 = 0.$$

The other double point is at $(0, 1, 1, -1)$.

If the center of projection be taken in π, the double line becomes tacnodal. The equation is

$$(x_1^2 + x_3 x_5)^2 - 41\, x_3^2 x_4 = 0.$$

$(0, 0, 1, 0)$ is a triple point on the tacnodal line.

From a point not on Γ the contour F_6 is of class 6, has a cuspidal c_6 and a cuspidal line intersecting in a triple point of F_6. It is focal surface for four (3, 3) congruences, one of which has the cuspidal line for focal line. If Γ has a double line, F_6 will have a double line intersecting c_6 in another triple point on the surface. One of the (3, 3) congruences has the new double line for focal line. If Γ has a double point, F_6 will be of class 4. It has four systems (3, 2). If Γ has both a double line and a double point, F_6 is of class 4, has a cuspidal line and an intersecting double line, and a double point. The cuspidal c_6 has two double points, d being tangent to a branch at one node, d' being tangent to a branch at the other.

If, however, the center of projection is in the plane π, the contour will be a ruled surface having a triple directrix, the three generators issuing from any

point lying in a plane containing the directrix. It has three cuspidal generators, and is contained in a special linear congruence.*

But if Γ has a double point, π lies on Γ, and the contour becomes a ruled quartic contained in a special linear congruence. From $(0, 0, 1, 1, 1)$ it becomes

$$(x_1 x_4 + x_2 x_5)^2 - 4(x_1^2 - x_2^2) x_1 x_2 = 0.$$

It has the symbol $[(21)\,111]$.

If Γ has both D and d', the corresponding contour

$$(x_1 x_4 + x_2 x_5)^2 - 4(x_1 + x_2) x_1^2 x_2 = 0$$

has the double generator $x_1 = 0$, $x_5 = 0$. Symbol $[(12)\,12]$. From $(0, 0, 1, 0, 0)$ we have an F_4 with a triple line. It has the symbol $[(31)\,11]$.

§ 5. *Two Double Lines of Second Species.*

26. Let $d\,(x_1 = 0,\ x_2 = 0,\ x_3 = 0)$ and $d'\,(x_1 = 0,\ x_3 = 0,\ x_4 = 0)$ be two lines on Γ, each of the second species. Along d, $x_3 = 0$ is a fixed tangent space, the other one describing a pencil projective with the points of d, the basis plane being $x_1 = 0$, $x_2 = 0$. Similarly, $x_3 = 0$ is fixed for points of d', the basis plane of the variable tangent space being $x_1 = 0$, $x_4 = 0$. The equation may be written

$$(a_1 x_1 + a_3 x_3) x_1^2 + h_3 x_1 x_2 x_3 + g\, x_1 x_3 + (f_1 x_1 + f_4 x_4) x_2 x_3 + (c_1 x_1 + c_3 x_3) x_3^2 = 0.$$

The tangent space $x_3 = 0$ cuts Γ in the plane $x_1 = 0$ counted three times. The section of Γ by $x_1 = 0$ is

$$x_3 (f_4 x_2 x_4 + c_3 x_3^2) = 0,$$

i. e., the fixed plane containing the double lines and a quadric cone containing d, tangent plane $x_1 = 0$, $x_2 = 0$, and containing d', tangent plane $x_1 = 0$, $x_4 = 0$. If $c_3 = 0$, the cone breaks up into these two tangent planes. The space $x_2 = kx_1$ cuts Γ in a quadric whose section with $x_1 = 0$, $x_2 = 0$ consists of d and the variable line $g + k f_4 x_4 + c_1 x_3 = 0$ passing through the fixed point $x_4 = 0$, $g + c_1 x_3 = 0$. Let $g \equiv x_5$. To avoid duplication of terms we may write

$$a_1 = h_3 = f_4 = c_1 = 1, \qquad f_2 = 0.$$

* This is included in type IX of my enumeration of sextic scrolls of genus one, JOURNAL, Vol. XXV (1903), p. 87. If Γ has d', F_6 is of genus zero. It is now type XXVIII of rational surfaces. See p. 74.

Γ has an additional double point at $(0, 1, 0, 0, -1)$, and can be generated by the two conjugate systems

$$\begin{aligned} &\alpha x_2 - \beta x_1 = 0, \\ &\alpha x_1 - \gamma x_3 = 0, \\ &\alpha (x_3 + x_5) + \beta (x_1 + x_4) - \gamma (x_1 + a_3 x_3) = 0; \end{aligned}$$

$$\begin{aligned} &\alpha x_2 + \beta x_1 + \gamma (x_3 + x_5) = 0, \\ &-\alpha x_1 + \gamma (x_1 + x_4) = 0, \\ &\beta x_3 + \gamma (x_1 + a_3 x_3) = 0. \end{aligned}$$

The first system cuts d, the second cuts d'. Each is $(1, 2)$.

27. The apparent contour from a point on Γ is an F_4 with two coplanar cuspidal lines. When the double point exists, it becomes the Plücker complex surface of symbol [33]. If the center of projection be taken in the plane $x_1 = 0$, $x_3 = 0$, the result is a cubic with 4 double points, with symbol [222]. From any point in either basis plane the contour consists of two quadrics touching each other along the projections of the cuspidal lines. The symbol is [(21)(11)1].

From $(1, 0, 0, 0, 0)$ not on Γ, the contour becomes a double plane and a surface of order 4 having a cuspidal conic. If $a_3 = 0$, the cuspidal conic reduces to two straight lines intersecting in a triple point on the surface at $(0, 1, 0, -1)$. The equation is

$$4 x_3 (x_2 + x_3 + x_5)^3 + 27 x_2^2 x_4^2 = 0.$$

28. The two basis planes may lie in the same space. The equation may now be written in the form

$$\Gamma \equiv x_1^2 x_5 + \lambda x_1 x_2^2 + x_2^3 + x_1 x_3 x_4 = 0,$$

$x_1 = 0$, $x_3 = 0$ being the basis plane for the variable spaces belonging to points of d, and $x_1 = 0$, $x_4 = 0$ being the plane for d'. From any point in $x_1 = 0$, not on Γ, the apparent contour is a quartic surface with a cuspidal conic. From $(0, 1, 0, 0, 0)$ it becomes two quadrics having two generators in common. From an arbitrary point not in $x_1 = 0$ the contour is a surface of order 6 having three cuspidal conics passing through two common points, at which they touch the same planes, and two cuspidal lines, the latter passing through the points of intersection of the conics and lying in the tangent planes.

From $(1, 0, 0, 0, 1)$ the surface is

$$4 (x_3 x_4 - 3 x_1^2)^3 + 27 (\lambda x_1 x_2^2 - 2 x_1^3 + \lambda x_2^3 + x_1 x_3 x_4)^2 = 0.$$

The cuspidal lines are $x_2=0$, $x_3=0$; $x_2=0$, $x_4=0$. The three cuspidal conics are cut from the cone $x_3x_4-3x_1^2=0$ by the planes $x_2=kx_1$, wherein

$$\lambda k^2(k+1)=-1.$$

If one of these conics be regarded as the absolute, a number of types of cyclides and other related surfaces can be obtained.

§ 6. *Depiction of* Γ *on* S_3.

29. Any cubic variety containing a plane can be birationally mapped on ordinary space. Let a plane π and a non-incident line d lie on Γ. From any point of S_4 can be drawn one line cutting d, π. Thus, any point P of Γ is uniquely associated with the point in S_3 in which the line through P cutting d, π cuts S_3. Since any six spaces in S_4 can be projected into any given ones not having an S_2 in common, we may define π by $x_1=0$, $x_2=0$; d by $x_3=0$, $x_4=0$, $x_5=0$ and S_3 by $x_1+x_2+x_3+x_4+x_5=0$. The equations of the line connecting $(\xi)=(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)$ on Γ with d, π are

$$\xi_2x_1-\xi_1x_2=0, \qquad \xi_4x_3-\xi_3x_4=0, \qquad \xi_3x_5-\xi_5x_3=0,$$

and the corresponding point in S_3 becomes

$$\tau x_1=-\xi_1(\xi_3+\xi_4+\xi_5), \qquad \tau x_2=-\xi_2(\xi_3+\xi_4+\xi_5),$$
$$\tau x_3=\xi_3(\xi_1+\xi_2), \qquad \tau x_4=\xi_4(\xi_1+\xi_2), \qquad \tau x_5=\xi_5(\xi_1+\xi_2).$$

The equation of Γ may be written in the form

$$\Gamma\equiv(a\xi_3+b\xi_4+c\xi_5)\xi_1-(a'\xi_3+b'\xi_4+c'\xi_5)\xi_2,$$

in which $a=\sum_{i=1}^{5}a_i\xi_i$, and similarly for the other terms. By means of the equations of the line we can now solve for ξ_i, giving the following results:

$$\begin{aligned}
\sigma\xi_1&=x_1[x_2\Sigma'-x_1\Sigma],\\
\sigma\xi_2&=x_2[\quad\text{``}\quad],\\
\sigma\xi_3&=x_3[A_1x_1^2+A_{12}x_1x_2+A_2'x_2^2],\\
\sigma\xi_4&=x_4[\qquad\text{``}\qquad],\\
\sigma\xi_5&=x_5[\qquad\text{``}\qquad],
\end{aligned}$$

wherein

$$\begin{aligned}
A_i&=a_ix_3+b_ix_4+c_ix_5,\\
A_{12}&=(a_2-a_1')x_3+(b_2-b_1')x_4+(c_2-c_1')x_5,\\
\Sigma&=A_3x_3+A_4x_4+A_5x_5,
\end{aligned}$$

with similar expressions for A_i', Σ_i'.

Any line of Γ which cuts both π and d will go into a point; one cutting d and not π, or π and not d will go into a straight line; one cutting neither will go into a conic. The line d and the plane π are principal elements in the depiction. The lines cutting two transversals of d, π can be depicted on a system of circles with common absolute points.

A linear transformation which leaves Γ invariant will define a Cremona transformation in S_3, and any other variety left invariant by the same transformation will define an invariant surface in S_3. Conversely, corresponding to any linear or Cremona group in S_3 there will be a group of birational transformations leaving Γ invariant.

An interesting application can be made to the varieties given in Nos. 19, 22, 23, 28, when we consider the one-parameter groups

$$\sigma \xi_1 = \xi_1', \quad \sigma \xi_2 = \xi_2', \quad \sigma \xi_3 = \xi_3', \quad \sigma \xi_4 = \xi_4' - \lambda \xi_2', \quad \sigma \xi_5 = \xi_5' + \lambda \xi_1'$$

under which Γ remains invariant. Any function of $\xi_1, \xi_2, \xi_3, \xi_1 \xi_4 + \xi_2 \xi_5$, together with Γ, will define a surface in S_4 which will become a surface in S invariant under a continuous Cremona group. But the resulting suface will contain only x_1, x_2, x_3 and hence defines a cone. From the theory of birational transformations of plane curves it now follows that the genus of this cone can not be greater than one, whatever function be chosen.

Another illustration is furnished by the variety

$$x_1^2 x_4 - x_2^2 x_5 + x_1 x_3 x_5 = 0,$$

a particular case of that considered in No. 19. It was shown by Fano* that it is invariant under the four-parameter continuous linear group defined by

$$\begin{aligned}
x_1' &= x_1, \\
x_2' &= \rho (x_2 + \alpha x_1), \\
x_3' &= \rho^2 (x_3 + 2 \alpha x_2 + (\alpha^2 - \beta) x_1), \\
x_4' &= \sigma \rho^2 (x_4 + \beta x_5), \\
x_5' &= \sigma x_5.
\end{aligned}$$

* "Sulle varietà algebriche dello spazio a quattro dimensioni ....," *Atti Ist. Venuto*, Ser. 7, Vol. VII (1896), pp. 1069–1103.

The corresponding birational transformation in $\Sigma x_i = 0$ becomes

$$\begin{aligned}
\tau x_1' &= -x_1 A,\\
\tau x_2' &= -\rho(x_2 + \alpha x_1) A,\\
\tau x_3' &= \rho^2 B x_3 (C - 2\alpha x_1 x_2 x_5 - (\alpha^2 - \beta) x_1^2 x_5),\\
\tau x_4' &= \sigma \rho^2 BC (x_4 + \beta x_5),\\
\tau x_5' &= \sigma x_5 BC.
\end{aligned}$$

$$\begin{aligned}
A &= \rho^2 x_3 C - 2\alpha\rho^2 x_1 x_2 x_3 x_5 - \rho^2(\alpha^2 - 1) x_1^2 x_3 x_5 + C\sigma(\rho^2 x_4 + (\rho^2\beta + 1) x_5),\\
B &= \rho x_2 + (\rho\alpha + 1) x_2,\\
C &= x_1^2 x_4 - x_2^2 x_5.
\end{aligned}$$

The pencil of quadric varieties $x_2^2 - x_1 x_3 = \lambda x_1^2$ and the pencil of spaces $x_1 = \mu x_5$ remain invariant under this group. In particular the form

$$(1 - \rho^2)(\xi_1^2 - \xi_1 \xi_3) - \rho^2 \beta \xi_1^2$$

is multiplied by ρ^2, and $\xi_1 \xi_5$ is multiplied by σ. Various functions of these two forms will therefore preserve their form when an appropriate relation between ρ, σ is given. Thus, if $\sigma = \rho^2$, we obtain the quartic surface

$$x_1^2 x_3 [(1 - \rho^2) x_4 - \rho^2 \beta x_5] + x_5 (x_1^2 x_4 - x_2^2 x_5) = 0$$

which has the three-parameter group. It is of type [2211]. This surface can be obtained by the method of Cremona.* Another form of quartic is obtained when $\sigma\rho^2 = 1$. When $\sigma^3 = \rho^2$, we obtained the surface of order 10

$$x_1^6 x_3^3 [(1 - \rho^2) x_4 - \rho^2 \beta x_5] = x_5 (x_1^2 x_4 - x_2^2 x_5)^3.$$

All these surfaces can be mapped birationally upon an arbitrary plane.

CORNELL UNIVERSITY, *December*, 1908.

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* "Rappresentatione piana di alcune superficie algebriche dotate di curve cuspidali," *Bologna Mem.*, Ser. 3, Vol. II (1872). See also Cremona's note in the *Göttinger Nachrichten*, 1871.